Continuous Time General Equilibrium

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An Endowment Economy in Discrete Time

- A unit mass of productive assets that produce consumption goods Y_t in each period, where the process follows an AR(1) process

$$\ln(Y_{t+1}) = \phi \ln(Y_t) + \sigma \varepsilon_{t+1}$$

- A continuum of households live in the economy. They can trade both the riskfree bonds and the productive asset in a fully competitive market. Objective function:

$$\max E[\sum_{t=0}^{\infty} \beta^t u(c_t)]$$

- Denote price of productive asset as P_t (after production) and interest rate as R_t.
- Question: Solve for the price of the productive asset and interest rate in a recursive equilibrium.

An Endowment Economy in Discrete Time

- Step 1: Determine state variables.
  - The same preference $\Rightarrow$ Only the aggregate matters $\Rightarrow$ Y_t is the only state variable. T or F?

An Endowment Economy in Discrete Time

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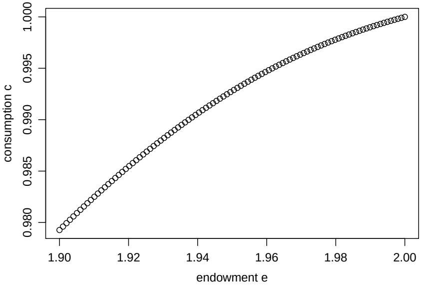
  - Counterexample: A simple two-period economy, with storage technology and initial endowment $\{e_i, i \in [0,1]\}$, and initial production $Y_0 = \sum_i e_i$. Set utility function as $u(c) = c - 1/c$.

$$\max_{c_i} \quad u(c_i) + \beta u(e_i - c_i) \Rightarrow 1 + \frac{1}{c_i^2} = \beta(1 + \frac{1}{(e_i - c_i)^2})$$



An Endowment Economy in Discrete Time

- Step 1: Determine state variables.
  - Let's assume $u(c) = c^{1-\gamma}/(1-\gamma)$, which will generate the property needed for aggregation. Other more complex utilities, including Epstein-Zin, will also work. Then the state variable is only Y_t.
  - Typical asset pricing exercise will stop here:

$$E_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{P_{t+1} + Y_{t+1}}{P_t}] = 1$$

$$E_t[\beta \frac{u'(c_{t+1})}{u'(c_t)} R_{f,t+1}] = 1$$

  - However, this is not enough for fully solving a general equilibrium.

An Endowment Economy in Discrete Time

- Step 2: Individual optimization problem.

$$V(w_t, Y_t) = \max_{x_t, c_t} \{u(c_t) + \beta E_t[V(w_{t+1}, Y_{t+1})]\}$$

$$s.t.$$

$$\begin{cases} w_{t+1} = w_t x_t \frac{P_{t+1}+Y_{t+1}}{P_t} + w_t(1-x_t)R_{f,t+1} - c_t \\ w_{t+1} \geq 0 \end{cases}$$

Property: x_t and c_t/w_t not dependent on $w_t \Rightarrow$ Aggregation.

An Endowment Economy in Discrete Time

- Step 2: Individual optimization problem.

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Property: x_t and c_t/w_t not dependent on $w_t \Rightarrow$ Aggregation.

- Step 3: Market clearing.
  - Total wealth equal to total assets:

$$W_t = P_t$$

  - Aggregate net borrowing is zero (bond market clearing):

$$1 - x_t = 0$$

  - Consumption equal to production:

$$C_t = Y_t$$

An Endowment Economy in Continuous Time

- A unit mass of productive assets that produce consumption goods $Y_t = \exp(\tilde{Y}_t)$, where the process follows an OU process

$$d\tilde{Y}_t = -\phi \tilde{Y}_t dt + \sigma dB_t$$

  Note: This is a continuous time analog to AR(1) process.

- A continuum of households live in the economy. They can trade both the riskfree bonds and the productive asset in a fully competitive market. Objective function:

$$\max E[\int_0^\infty e^{-\rho t} u(c_t)]$$

An Endowment Economy in Continuous Time

- Step 1: Determine state variables.
  - Under a homogeneous economy (all agents have exactly the same endowment to start with), or $u(c_t)$ belongs to CRRA (or other appropriate) utility functions, the only state variable is Y_t.
  - Extra step: The returns of assets in a continuous stochastic environment is not immediately clear. Conjecture

$$\frac{dP_t}{P_t} = \mu_{P,t} dt + \sigma_{P,t} dB_t$$

Trick: All processes will be within the same class of Ito's processes. The largest extensible class is semi-martingales, where both jump and diffusions appear. Then the return of productive asset is

$$\frac{Y_t dt + dP_t}{P_t} = (\frac{Y_t}{P_t} + \mu_{P,t}) dt + \sigma_{P,t} dB_t$$

An Endowment Economy in Continuous Time

- Step 2: Individual optimization problem.

$$V(w_t, Y_t) = \max_{x_t, c_t} E_t[\int_t^\infty e^{-\rho t} u(c_s) ds]$$

$$s.t.$$

$$\begin{cases} dw_t = w_t x_t \frac{Y_t dt + dP_t}{P_t} + w_t(1 - x_t) r_t dt - c_t dt \\ w_t \geq 0 \end{cases}$$

- Step 3: Market clearing.
  - Total wealth equal to total assets (budget constraint):

$$W_t = P_t$$

  - Aggregate net borrowing is zero (bond market clearing):

$$1 - x_t = 0$$

  - Productive asset market clearing is implied by the above two from Walras law.
  - Consumption equal to production:

$$C_t = Y_t$$

An Endowment Economy in Continuous Time

- The problem is more tractable under continuous time with CRRA utility. With log utility, we don't need to solve ODEs. Under log utility, agents behave as if "myopic",

$$c_t = \rho w_t$$

$$x_t = \frac{Y_t/P_t + \mu_{P,t} - r_t}{\sigma^2_{P,t}}$$

- Market clearing implies

$$Y_t = \rho P_t$$

$$Y_t/P_t + \mu_{P,t} - r_t = \sigma^2_{P,t}$$

- Why we don't have enough equations?
  - Special technique in continuous time: Differential over market clearing condition. Match both dt and dB_t.

$$Y_t \sigma dB_t + Y_t \left(-\phi \ln(Y_t) + \frac{1}{2}\sigma^2 \right) dt = \rho P_t \mu_{P,t} dt + \rho P_t \sigma_{P,t} dB_t$$

$$\Rightarrow \begin{cases} \sigma_{P,t} = \sigma \\ \mu_{P,t} = \frac{-\phi \ln(Y_t) + \sigma^2/2}{\rho P_t} \end{cases}$$

An Endowment Economy in Continuous Time

- Summary of solutions

$$
\begin{aligned}
P_t &= Y_t/\rho \\
\sigma_{P,t} &= \sigma \\
\mu_{P,t} &= \frac{-\phi \ln(Y_t) + \sigma^2/2}{Y_t} \\
r_t &= \rho - \sigma^2 + \frac{-\phi \ln(Y_t) + \sigma^2/2}{Y_t}
\end{aligned}
$$

- Key advantages of continuous time:
  (1) Easier to solve for portfolio choice problems.
  (2) Simpler representation and calculation (Ito calculus) of stochastic processes.

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Portfolio Choice under Log Utility

- Key simplification: In a competitive market, each agent takes price processes as fully exogenous.
- Asset return (could be a vector)

$$dR_t = \mu(s_t)dt + \sigma(s_t)dB_t$$

  where s_t is a vector of state variables that also follows a Ito process where Ito calculus is applicable.
- An individual with log utility and pure consumption portfolio choice problem:

$$\max E[\int_0^\infty e^{-\rho t} \log(c_t)]$$
$$s.t.$$
$$\begin{cases} \frac{dw_t}{w_t} = x_t \cdot dR_t + (1 - \mathbf{1} \cdot x_t)r_t dt - \frac{c_t}{w_t} dt \\ w_t \geq 0 \end{cases}$$

- Value function denoted as $V(w_t, s_t)$.

Portfolio Choice under Log Utility

- Key solution technique: HJB equation + conjecture verification.

- Suppose $\{c_s^*, x_s^*, s \geq t\}$ are optimal starting from $w_t = 1$, then $\{w_t c_s^*, x_s^*, s \geq t\}$ are optimal starting from w_t.
  - Immediate implication:

$$V(w_t, s_t) = E_t[\int_t^\infty e^{-\rho(s-t)} \log(c_s) ds]$$

$$= E_t[\int_t^\infty e^{-\rho(s-t)} \log(w_t c_s^*) ds] = \frac{1}{\rho} \log(w_t) + v(s_t)$$

    for some $v(\cdot)$, where c_t^* is the path starting from $w_t = 1$.
  - Proof: counterarguments.

- With the separable value function, we can now derive the HJB equation and portfolio choice problems explicitly.

Portfolio Choice under Log Utility

- HJB equation derivation:

$$V(w_t, s_t) \approx \max_{c_t, x_t} \left\{ \log(c_t)dt + E_t[e^{-\rho dt} V(w_{t+dt}, s_{t+dt})] \right\}$$

$$\approx \max_{c_t, x_t} \{ \log(c_t)dt + (1 - \rho dt) E_t[V(w_{t+dt}, s_{t+dt})] \}$$

- By Ito's calculus,

$$E_t[V(w_{t+dt}, s_{t+dt})] = V(w_t, s_t) + E_t[V_w{}'(w_t, s_t) dw_t + \frac{1}{2} V_w{}''(w_t)(dw_t)^2$$

$$+ V_s{}'(w_t, s_t) ds_t + \frac{1}{2} V_{ss}{}'(w_t, s_t)(ds_t)^2 + V_{sw}{}''(w_t, s_t) dw_t ds_t]$$

Here the separability of value function helps to reduce the above to

$$E_t[V(w_{t+dt}, s_{t+dt})] = V(w_t, s_t) + E_t[\frac{1}{\rho} \frac{dw_t}{w_t} - \frac{1}{2} \frac{1}{\rho} (\frac{dw_t}{w_t})^2$$

$$+ v_s{}'(s_t) ds_t + \frac{1}{2} v_{ss}{}'(s_t)(ds_t)^2]$$

Portfolio Choice under Log Utility

- Optimization:

$$\max_{c_t, x_t} \left\{ \log(c_t)dt + E_t[\frac{1}{\rho}\frac{dw_t}{w_t} - \frac{1}{2\rho}(\frac{dw_t}{w_t})^2] \right\}$$

$$\Rightarrow \max_{c_t, x_t} \left\{ \rho \log(c_t) - \frac{c_t}{w_t} + x_t \cdot (\mu_t - r_t) - \frac{1}{2} x_t^T \sigma_t \sigma_t^T x_t \right\}$$

$$\Rightarrow c_t = \rho w_t, \qquad x_t = (\sigma_t \sigma_t^T)^{-1}(\mu_t - r_t)$$

- Important: The state variable s_t could almost follow any process without affect the portfolio choice problems of a log-utility agent. Easy to embed into a general equilibrium problem.

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The Model Setup in Problem Set 3

- A unit mass of bankers and households. Bankers solve

$$E_s[\int_s^\infty e^{-\rho(t-s)} \ln(c_t) dt]$$

and households solve

$$E_s[\int_s^\infty e^{-\rho^h(t-s)} \ln(c_t^h) dt]$$

- Output from per unit productive asset

$$\frac{dY_t}{Y_t} = gdt + \sigma_Y dZ_t$$

Only bankers can hold the productive asset. No bank equity issuance to households.

- Household labor income (why we need labor income?)

$$\frac{dL_t}{L_t} = gdt + \sigma_L dZ_t$$

Equilibrium Solution

- Step 1: Determine state variables.

  Y_t, L_t, and banker wealth share w_t (or leverage).

  Q: What if we have a third type of agents?

- Step 2: Individual optimization problem.

$$x_t = \frac{Y_t/P_t + \mu_{P,t} - r_t}{\sigma_{P,t}^2}$$

$$c_t^b = \rho w_t$$

$$c_t^h = \rho^h w_t^h$$

Equilibrium Solution

- Step 3: Market clearing.
  - Consumption goods clearing.

$$C_t^b + C_t^h = Y_t + L_t$$

$$\Rightarrow \rho W_t^b + \rho^h W_t^h = Y_t + L_t$$

  - Risky asset investment clearing.

$$x_t W_t^b = P_t$$

  - Bond market clearing, which is implied by Walras law and wealth identity

$$W_t^b + W_t^h = P_t$$

- Special trick: Differentiating on the aggregate equation to match dynamics.

$$d(\rho W_t^b + \rho^h W_t^h) = d(Y_t + L_t)$$

Equilibrium Solution

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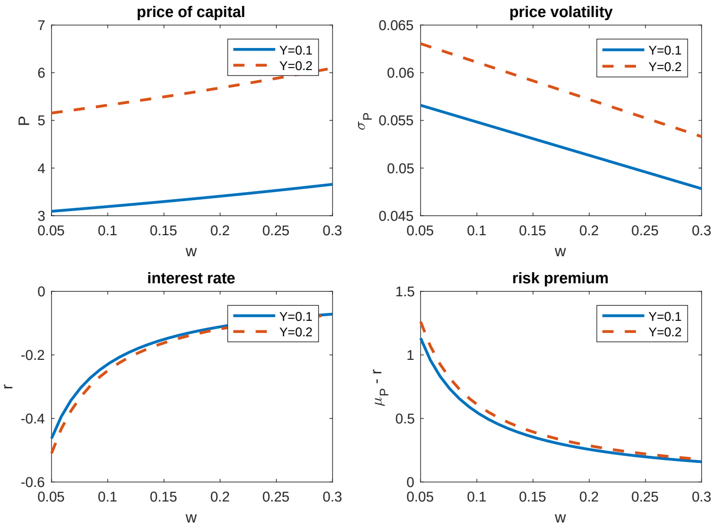
Equilibrium Solution



What if Households Can Directly Invest in Capital?